Lecture 2:
Asymptotic Analysis of Algorithms

Goodrich & Tamassia, Chapter 4

Overview

- Motivation
- Definition of Running Time
- Classifying Running Time
- Asymptotic Notation & Proving Bounds
- Algorithm Complexity vs Problem Complexity

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The Importance of Analyzing Run Time

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Thu, 26 Jul 2001 00:50:03 +0300

Subject: New results on WEP (via Matt Blaze)

WEP is the security protocol used in the widely deployed IEEE 802.11 wireless LAN's. This protocol received a lot of attention this year, and several groups of researchers have described a number of ways to bypass its security.

Attached you will find a new paper which describes a truly practical direct attack on WEP's cryptography. It is an **extremely powerful attack** which can be applied even when WEP's RC4 stream cipher uses a 2048 bit secret key (its maximal size) and 128 bit IV modifiers (as proposed in WEP2). The attacker can be a completely passive eavesdropper (i.e., he does not have to inject packets, monitor responses, or use accomplices) and thus his existence is essentially undetectable. It is a pure known-ciphertext attack (i.e., the attacker need not know or choose their corresponding plaintexts). **After scanning several hundred thousand packets, the attacker can completely recover the secret key and thus decrypt all the ciphertexts. The running time of the attack grows linearly instead of exponentially with the key size, and thus it is negligible even for 2048 bit keys.**

Adi Shamir

Source: The Risks Digest (catless.ncl.ac.uk/Risks)

The Importance of Analyzing Run Time

<Monty Solomon <monty@roscom.com>>

Sat, 31 May 2003 10:22:56 -0400

Denial of Service via Algorithmic Complexity Attacks

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We present a new class of low-bandwidth denial of service attacks that exploit algorithmic deficiencies in many common applications' data structures. **Frequently used data structures have ``average-case'' expected running time that's far more efficient than the worst case. For example, both binary trees and hash tables can degenerate to linked lists with carefully chosen input.** We show how an attacker can effectively compute such input, and we demonstrate attacks against the hash table implementations in two versions of Perl, the Squid web proxy, and the Bro intrusion detection system. **Using bandwidth less than a typical dialup modem, we can bring a dedicated Bro server to its knees**; after six minutes of carefully chosen packets, our Bro server was dropping as much as 71% of its traffic and consuming all of its CPU. We show how modern universal hashing techniques can yield performance comparable to commonplace hash functions while being provably secure against these attacks.

Source: The Risks Digest (catless.ncl.ac.uk/Risks)

The Purpose of Asymptotic Analysis

- To estimate how long a program will run.
- To estimate the largest input that can reasonably be given to the program.
- To compare the efficiency of different algorithms.
- To help focus on the parts of code that are executed the largest number of times.
- To choose an algorithm for an application.

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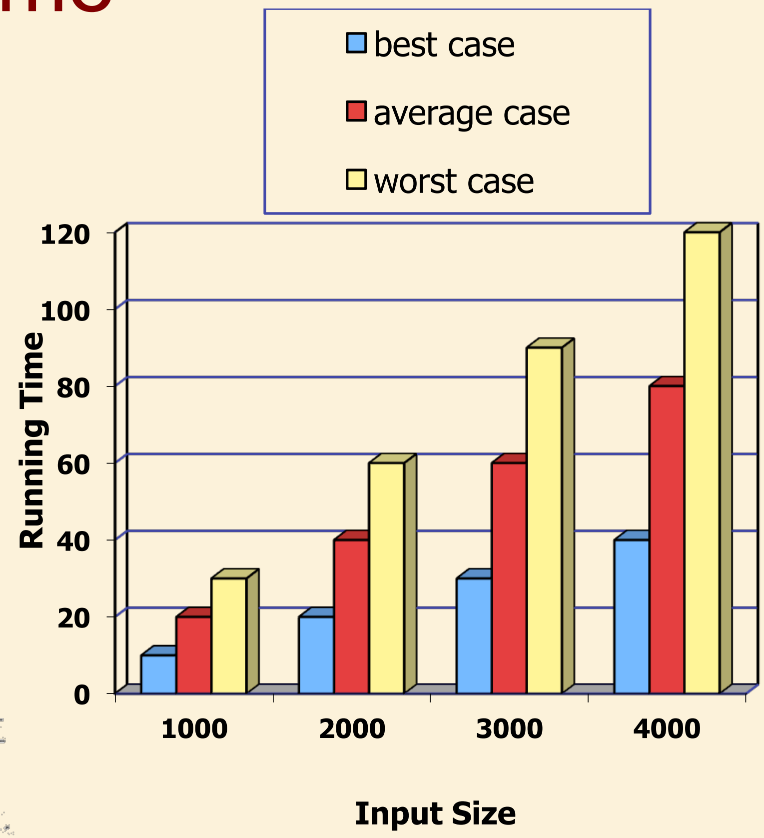
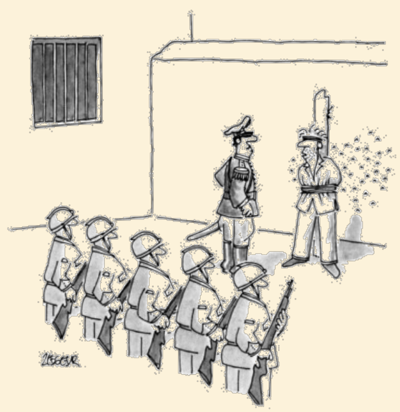
Running Time

- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically grows with the input size ***n***.
- Average case time is often difficult to determine.
- We focus on the worst case running time.
  - Easier to analyze
  - Reduces risk





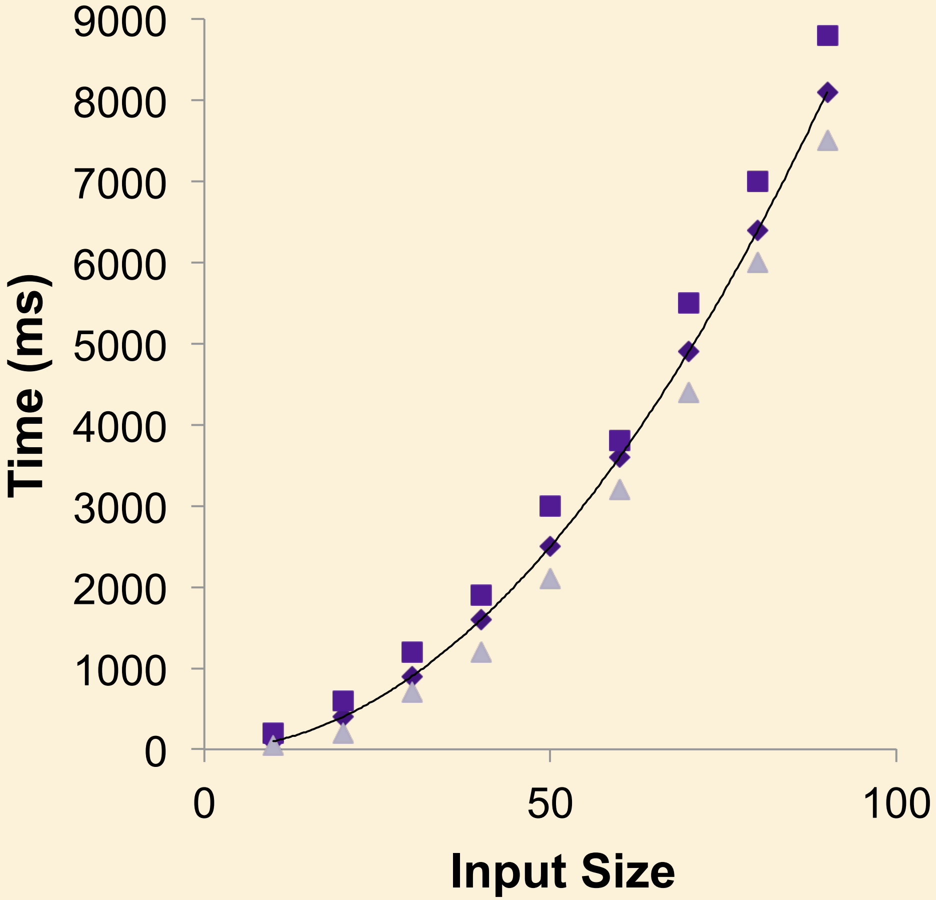
"Level with me, Colonel. What kind of worst-case scenario are we talking about here?"

Asymptotic Analysis

- In this context ‘asymptotic’ simply means ‘for large input size’.
- We don’t worry about small inputs – these will be easy.
- Rather we care about how run time will ultimately increase as the input size ***n*** gets larger and larger.
- This will tend to limit the maximum size of input the algorithm can handle.

Experimental Studies

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition
- Use a method like System.currentTimeMillis() to get an accurate measure of the actual running time
- Plot the results



Limitations of Experiments

- It is necessary to implement the algorithm, which may be difficult
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used

Theoretical Analysis

- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, ***n***.
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Assumed to take a constant amount of time

- Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method

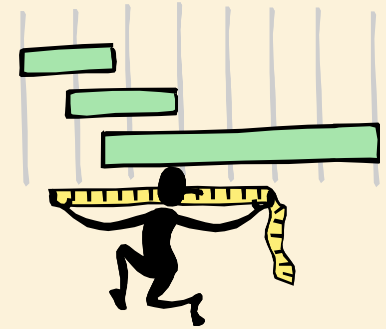
Counting Primitive Operations

- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

Algorithm ***arrayMax***(***A***, ***n***)

	# operations
currentMax ← ***A***[0]	?
for ***i*** ← 1 **to** ***n*** - 1 **do**	?
if ***A***[*i*] > ***currentMax*** **then**	?
currentMax ← ***A***[*i*]	?
return ***currentMax***	?
Total	?

Estimating Running Time



- Algorithm ***arrayMax*** executes $6n - 1$ primitive operations in the worst case. Define:

  a = Time taken by the fastest primitive operation

  b = Time taken by the slowest primitive operation

- Let $T(n)$ be worst-case time of ***arrayMax***. Then

$$a(6n-1) \le T(n) \le b(6n-1)$$

- Hence, the running time $T(n)$ is bounded by two linear functions

Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects $T(n)$ by a constant factor, but
  - Does not qualitatively alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm ***arrayMax***

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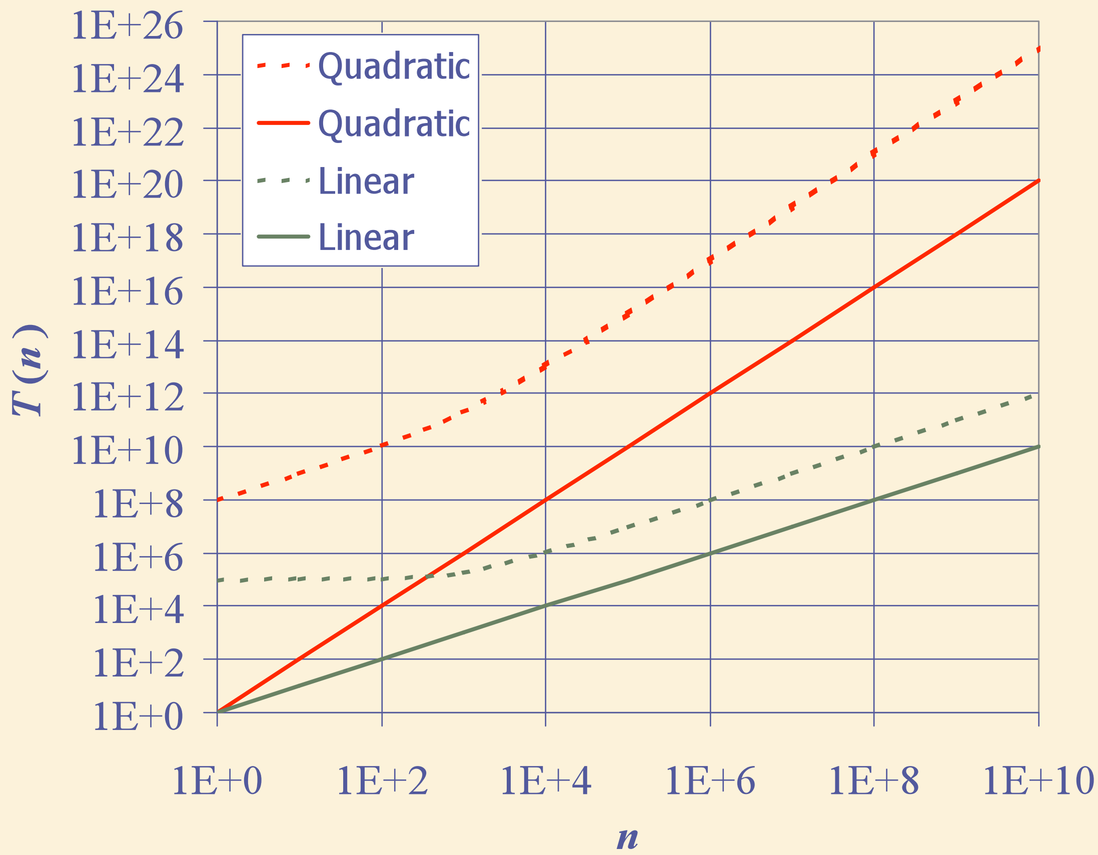
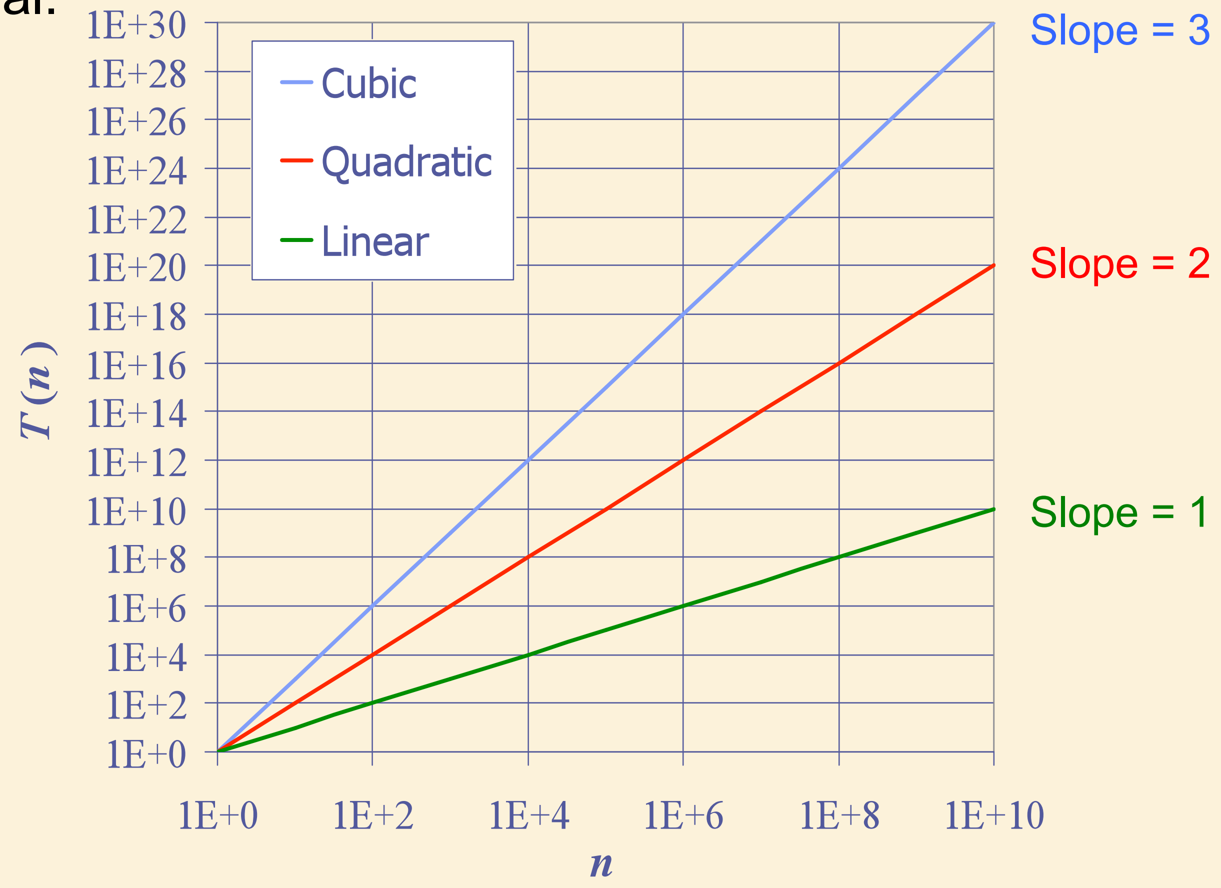
Constant Factors

- On a logarithmic scale, the asymptotic growth rate is not affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2 \boldsymbol{n} + 10^5$ is a linear function
  - $10^5 \boldsymbol{n}^2 + 10^8 \boldsymbol{n}$ is a quadratic function



Polynomial Growth

- Many algorithms that we encounter will have polynomial growth
- In a log-log chart, the asymptotic slope of the line corresponds to the order of the polynomial.



We will follow the convention that $\log n \equiv \log_2 n$.

Seven Important Functions

- Seven functions that often appear in algorithm analysis:
  - Constant ≈ 1
  - Logarithmic ≈ $\log n$
  - Linear ≈ n
  - N-Log-N ≈ $n \log n$
  - Quadratic ≈ n^2
  - Cubic ≈ n^3
  - Exponential ≈ 2^n
- Although the detailed expression for run time may be complicated, most algorithms we will encounter can be mapped to one of these simple categories.

Classifying Functions

	n			
$T(n)$	10	100	1,000	10,000
$\log n$	3	6	9	13
$n^{1/2}$	3	10	31	100
n	10	100	1,000	10,000
$n \log n$	30	600	9,000	130,000
n^2	100	10,000	10^6	10^8
n^3	1,000	10^6	10^9	10^{12}
2^n	1,024	10^{30}	10^{300}	10^{3000}

Note: The universe is estimated to contain ~10^{80} particles.

Let’s practice classifying functions

Which are more alike?

n^{1000} $\qquad$ n^2 $\qquad$ 2^n

Which are more alike?

$$\underbrace{n^{1000} \qquad n^2}_{\text{Polynomials}} \qquad 2^n$$

Which are more alike?

$1000n^2$ $\quad 3n^2 \quad$ $2n^3$

Which are more alike?

$\underbrace{1000n^2 \qquad 3n^2}_{\text{Quadratic}} \qquad 2n^3$

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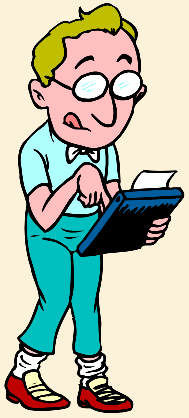
Some Math to Review



- Summations
- Logarithms and Exponents
- Existential and universal operators
- Proof techniques

- **existential and universal operators**

$$\exists g \forall b \; Loves(b, g)$$

$$\forall g \exists b \; Loves(b, g)$$

- **properties of logarithms:**

$\log_b(xy) = \log_b x + \log_b y$

$\log_b (x/y) = \log_b x - \log_b y$

$\log_b x^a = a\log_b x$

$\log_b a = \log_x a / \log_x b$

- **properties of exponentials**:

$a^{(b+c)} = a^b a^c$

$a^{bc} = (a^b)^c$

$a^b / a^c = a^{(b-c)}$

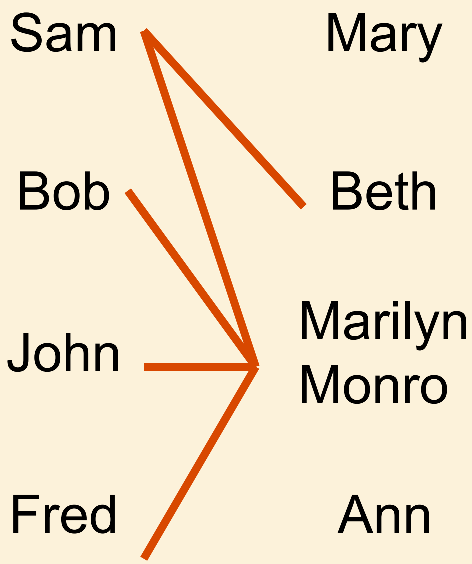
$b = a^{\log_a b}$

$b^c = a^{c*\log_a b}$

Understand Quantifiers!!!

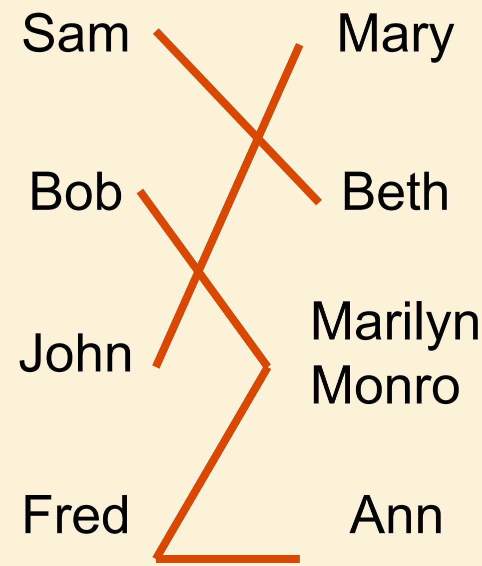
$$\exists g, \forall b, loves(b,g)$$

One girl



$$\forall g, \exists b, loves(b,g)$$

Could be a separate girl for each boy.



Asymptotic Notation
($\Omicron, \Omega, \Theta$ and all of that)

- The notation was first introduced by number theorist Paul Bachmann in 1894, in the second volume of his book *Analytische Zahlentheorie* ("analytic number theory").
- The notation was popularized in the work of number theorist Edmund Landau; hence it is sometimes called a Landau symbol.
- It was popularized in computer science by Donald Knuth, who (re)introduced the related Omega and Theta notations.
- Knuth also noted that the (then obscure) Omega notation had been introduced by Hardy and Littlewood under a slightly different meaning, and proposed the current definition.

Source: Wikipedia

Asymptotic Notation
($\mathrm{O}, \Omega, \Theta$ and all of that)

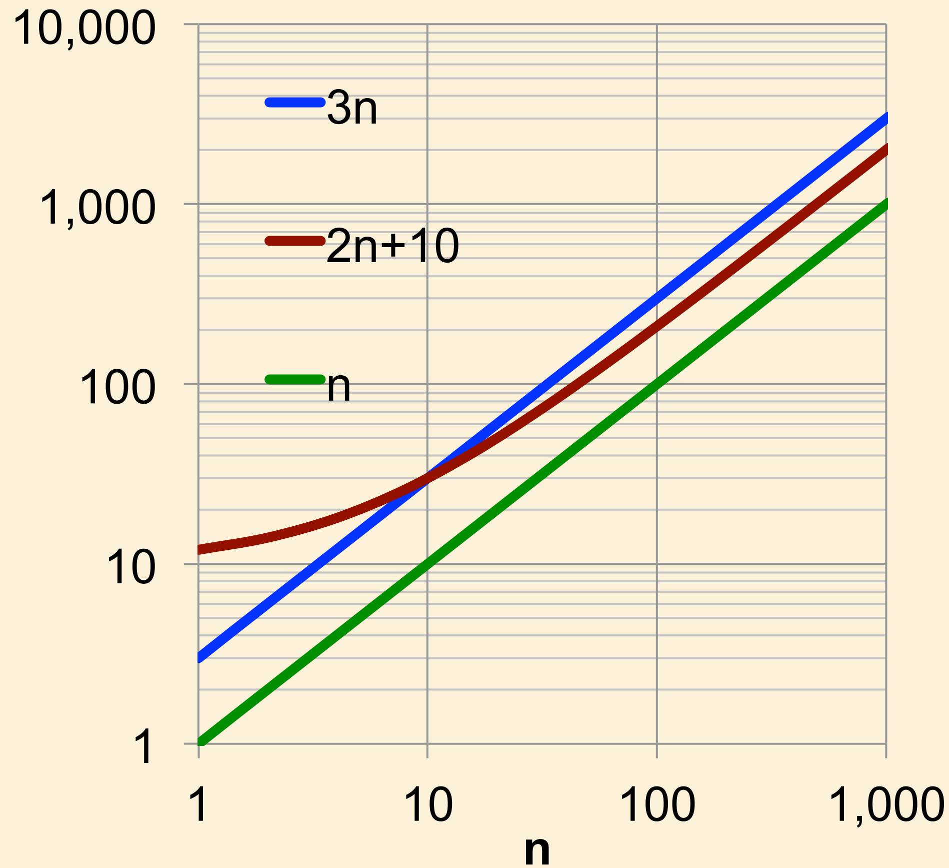
- Our primary use of this notation is to state and prove upper and lower asymptotic bounds on run time *T(n)*.
- However the notation applies to the growth of arbitrary functions *f(n)*.

Big-Oh Notation

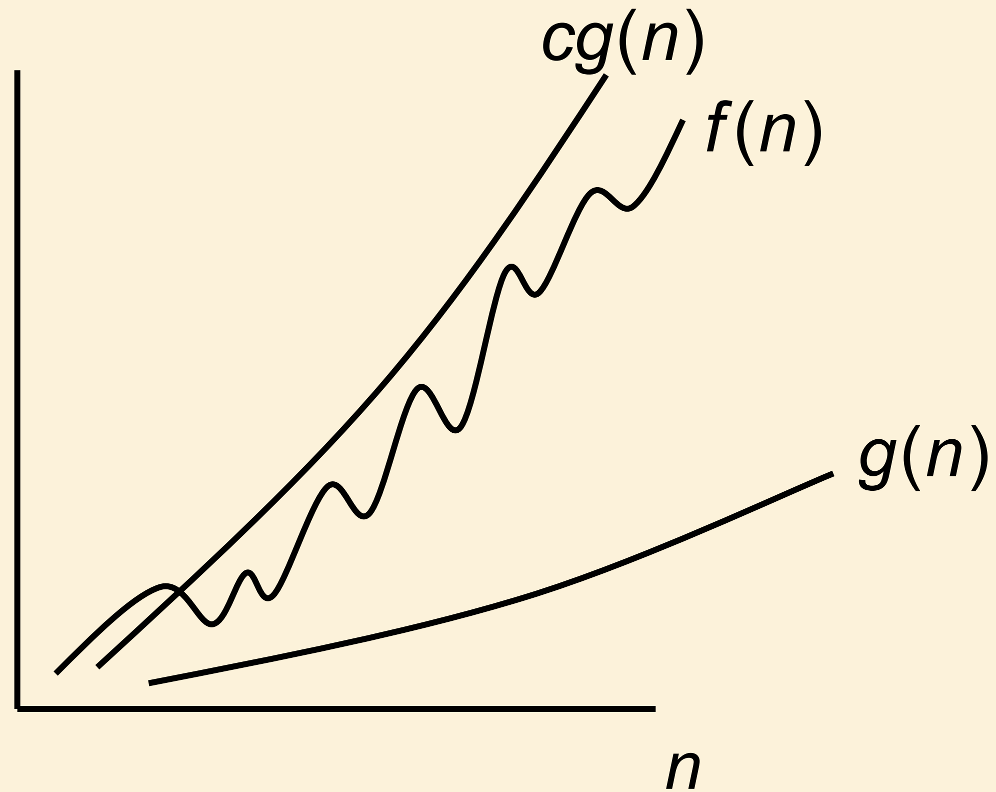
- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that

  $f(n) \le cg(n)$ for $n \ge n_0$

- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \le cn$
  - $(c - 2)\, n \ge 10$
  - $n \ge 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$



Definition of "Big Oh"

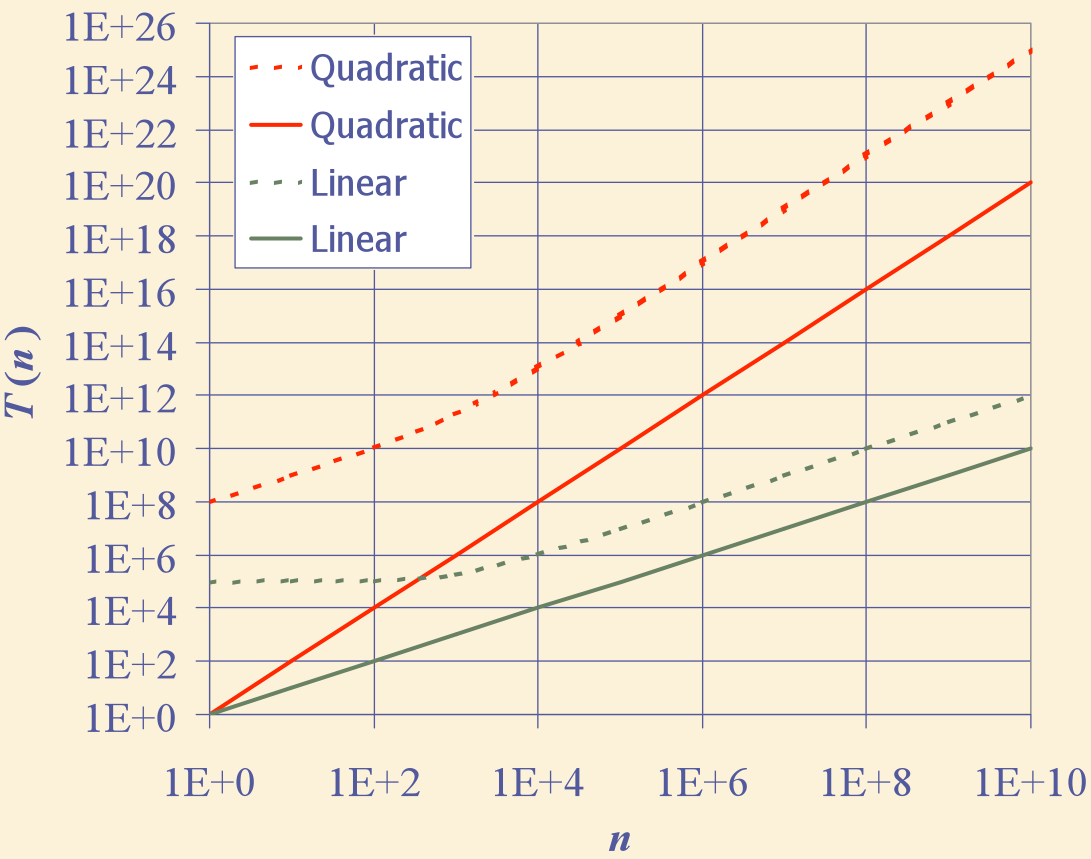


$$f(n) \in O(g(n))$$

$$\exists c, n_0 > 0 : \forall n \geq n_0, f(n) \leq cg(n)$$

Big-Oh Example

- Example: the function n^2 is not $O(n)$
  - $n^2 \leq cn$
  - $n \leq c$
  - The above inequality cannot be satisfied since c must be a constant



More Big-Oh Examples

- $7n-2$

  $7n-2$ is $O(n)$

  need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq c \cdot n$ for $n \geq n_0$

  this is true for $c = 7$ and $n_0 = 1$

- $3n^3 + 20n^2 + 5$

  $3n^3 + 20n^2 + 5$ is $O(n^3)$

  need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq c \cdot n^3$ for $n \geq n_0$

  this is true for $c = 5$ and $n_0 = 20$

- $3 \log n + 5$

  $3 \log n + 5$ is $O(\log n)$

  need $c > 0$ and $n_0 \geq 1$ such that $3 \log n + 5 \leq c \cdot \log n$ for $n \geq n_0$

  this is true for $c = 4$ and $n_0 = 32$

Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

	$f(n)$ is $O(g(n))$	$g(n)$ is $O(f(n))$
$g(n)$ grows more	Yes	No
$f(n)$ grows more	No	Yes
Same growth	Yes	Yes

Big-Oh Rules

- If $f(n)$ is a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- We generally specify the tightest bound possible
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Asymptotic Algorithm Analysis

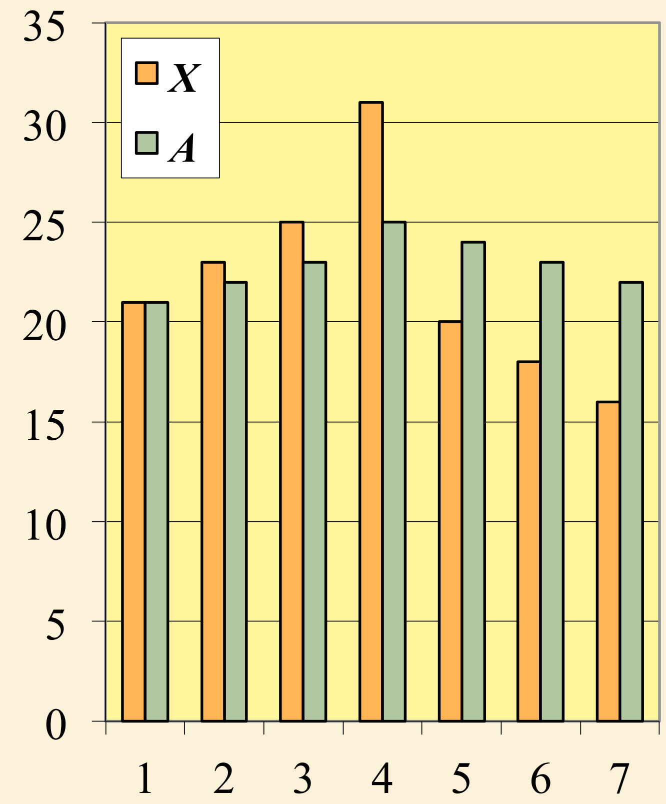
- The asymptotic analysis of an algorithm involves finding the running time in big-Oh notation
- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We determine that algorithm ***arrayMax*** executes at most $6n - 1$ primitive operations
  - We say that algorithm ***arrayMax*** "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

Computing Prefix Averages

- We further illustrate asymptotic analysis with two algorithms for prefix averages
- The $\boldsymbol{i}$-th prefix average of an array $\boldsymbol{X}$ is the average of the first $(\boldsymbol{i}+1)$ elements of $\boldsymbol{X}$:

$$\boldsymbol{A}[\boldsymbol{i}] = (\boldsymbol{X}[0] + \boldsymbol{X}[1] + \ldots + \boldsymbol{X}[\boldsymbol{i}])/(i+1)$$

- Computing the array $\boldsymbol{A}$ of prefix averages of another array $\boldsymbol{X}$ has applications to financial analysis, for example.



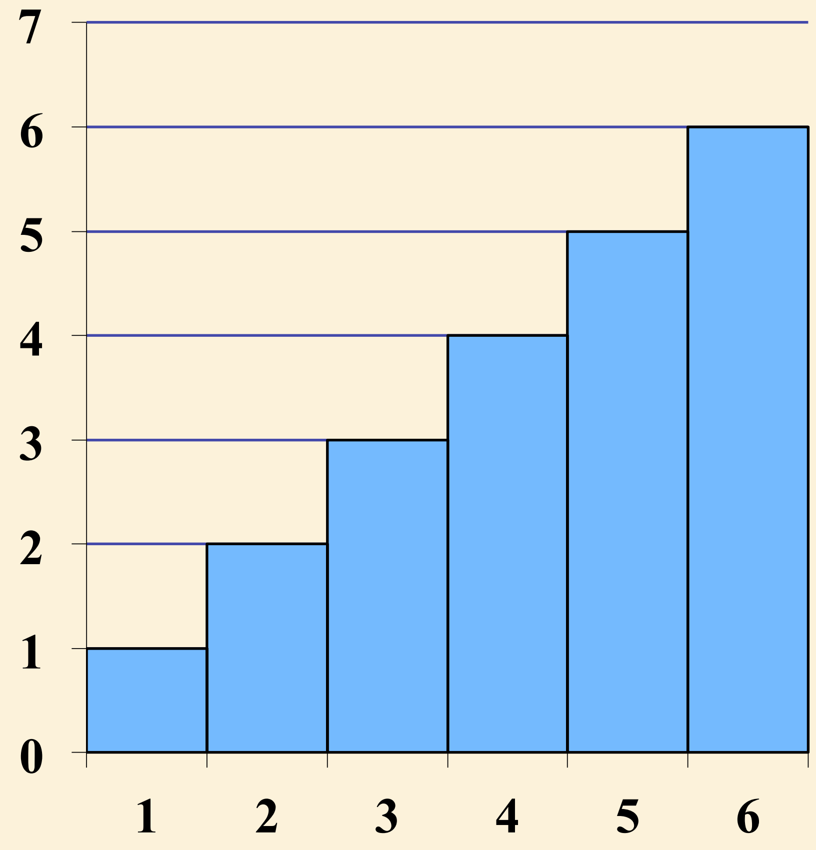
Prefix Averages (v1)

- The following algorithm computes prefix averages by applying the definition

Algorithm ***prefixAverages1*** (X, n)	
Input array X of n integers	
Output array A of prefix averages of X	#operations
$A \leftarrow$ new array of n integers	n
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$s \leftarrow X[0]$	n
for $j \leftarrow 1$ **to** i **do**	$1 + 2 + \ldots + (n - 1)$
$s \leftarrow s + X[j]$	$1 + 2 + \ldots + (n - 1)$
$A[i] \leftarrow s / (i + 1)$	n
return A	1

Arithmetic Progression

- The running time of ***prefixAverages1*** is $O(1+2+\ldots+n)$
- The sum of the first n integers is $n(n+1)/2$
  - There is a simple visual proof of this fact
- Thus, algorithm ***prefixAverages1*** runs in $O(n^2)$ time



Prefix Averages (v2)

- The following algorithm computes prefix averages efficiently by keeping a running sum

Algorithm *prefixAverages2*(X, n)	
Input array X of n integers	
Output array A of prefix averages of X	#operations
$A \leftarrow$ new array of n integers	n
$s \leftarrow 0$	1
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$s \leftarrow s + X[i]$	n
$A[i] \leftarrow s / (i + 1)$	n
return A	1

- Algorithm ***prefixAverages2*** runs in $O(n)$ time

Relatives of Big-Oh

- **Big-Omega**
  - f(n) is Ω(g(n)) if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$

- **Big-Theta**
  - f(n) is Θ(g(n)) if there are constants $c_1 > 0$ and $c_2 > 0$ and an integer constant $n_0 \geq 1$ such that $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for $n \geq n_0$

Intuition for Asymptotic Notation

Big-Oh

- f(n) is O(g(n)) if f(n) is asymptotically **less than or equal** to g(n)

big-Omega

- f(n) is Ω(g(n)) if f(n) is asymptotically **greater than or equal** to g(n)

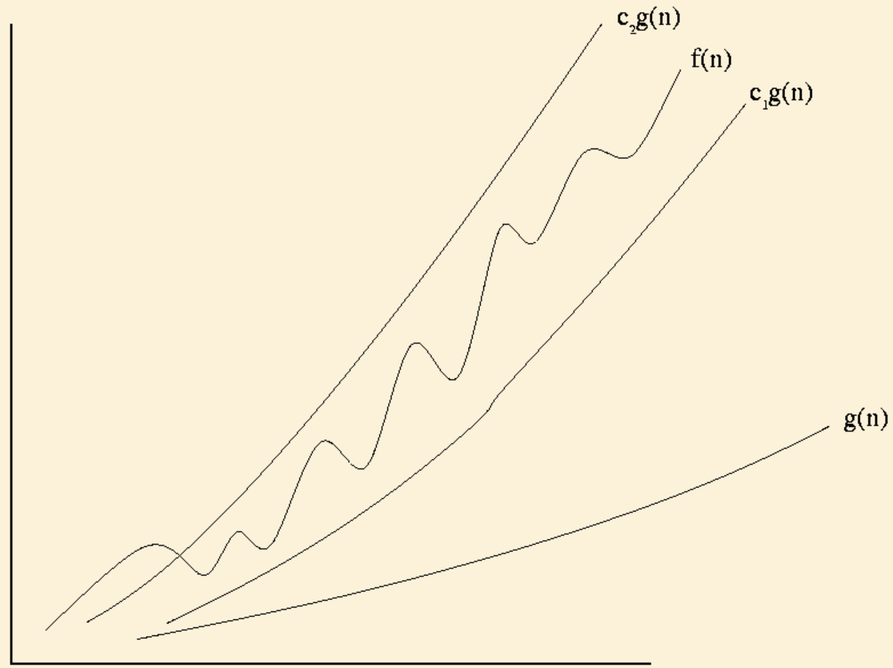
big-Theta

- f(n) is Θ(g(n)) if f(n) is asymptotically **equal** to g(n)

Note that $f(n) \in \Theta\left(g(n)\right) \equiv \left(f(n) \in O\left(g(n)\right) \text{ and } f(n) \in \Omega\left(g(n)\right)\right)$

Definition of Theta

$$f(n) = \theta(g(n))$$



$$\exists c_1, c_2, n_0 > 0 : \forall n \geq n_0, \underbrace{c_1 g(n) \leq f(n) \leq c_2 g(n)}_{}$$

f(n) is sandwiched between $c_1g(n)$ and $c_2g(n)$

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Time Complexity of an Algorithm

The time complexity of an algorithm is the *largest* time required on *any* input of size n. (Worst case analysis.)

- $O(n^2)$: For any input size $n \geq n_0$, the algorithm takes no more than cn^2 time on every input.
- $\Omega(n^2)$: For any input size $n \geq n_0$, the algorithm takes at least cn^2 time on at least one input.
- $\theta\ (n^2)$: Do both.

What is the height of tallest person in the class?

Bigger than this?

Need to find only one person who is taller

Smaller than this?

Need to look at every person

Time Complexity of a Problem

The time complexity of a problem is the time complexity of the *fastest* algorithm that solves the problem.

- $O(n^2)$: Provide an algorithm that solves the problem in no more than this time.
  - Remember: for every input, i.e. worst case analysis!
- $\Omega(n^2)$: Prove that no algorithm can solve it faster.
  - Remember: only need one input that takes at least this long!
- $\theta(n^2)$: Do both.

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